

HODGE THEORY AND  $A_\infty$  STRUCTURES ON COHOMOLOGY

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ABSTRACT. We use Hodge theory and a construction of Merkulov to construct  $A_\infty$  structures on de Rham cohomology and Dolbeault cohomology.

Hodge theory is a powerful tool in differential geometry. Classically, it can be used to identify the de Rham cohomology of a closed oriented Riemannian manifold with the space of harmonic forms on it as vector spaces. The wedge product on differential forms provides an algebra structure on the de Rham cohomology. By virtue of being isomorphic to the de Rham cohomology, the space of harmonic forms has an induced associative multiplication. However, the wedge product of two harmonic forms may not be harmonic. One needs to define a multiplication of two harmonic forms by taking the harmonic part of their wedge product, and then show that this multiplication is indeed associative and can be identified with the wedge product on the de Rham cohomology. In this paper, we show that one can actually take advantage of this awkward situation to construct higher multiplications and define a structure of  $A_\infty$  algebra on the space of harmonic forms.

Originally,  $A_\infty$  structures were introduced by Stasheff [7, 8] in 1963 in the study of  $H$  spaces. Together with various cousins, they have appeared in the last couple of decades in many places in Mathematics and Mathematical Physics. In particular, infinite structures are very useful in the formulation of mirror symmetry. For example, Fukaya [2] constructed  $A_\infty$  categories from symplectic manifolds, which is used in the formulation of homological mirror symmetry by Kontsevich [4]. For Calabi-Yau manifold, where the notion of mirror symmetry was originally conceived [9, 3], recent formulations of the mirror symmetry use the notion of Frobenius manifolds introduced by Dubrovin [1]. As pointed out in Manin [5], a formal Frobenius manifold structure on a vector space with a nondegenerate pairing is equivalent to a cyclic  $Comm_\infty$ -structure on it. The popular theory of quantum cohomology provides construction of formal Frobenius manifold structures on de Rham cohomology of symplectic manifolds. The appearance of infinite algebra structures in the theory of quantum cohomology and mirror symmetry indicates the importance of the study of infinite algebras in differential geometry.

Our construction is based on a recent paper of Merkulov [6], where he gave a nice construction of  $A_\infty$  algebra. Together with Hodge theory of closed Kähler manifold, he constructed an  $A_\infty$  structure on a subcomplex of the de Rham complex which contains the harmonic forms. Similar constructions can be carried out for the deformation complex of Calabi-Yau manifolds. The simple observation of this paper is that Merkulov's construction can be used to give a construction of  $A_\infty$  structure on the space of harmonic forms of any oriented closed Riemannian manifold. Similarly, Hodge theory of the deformation complex of any closed complex  $M$  manifold can be used to construct an  $A_\infty$  structure on  $H^{-*,*}(M)$ . Similar constructions can

be carried out for Dolbeault cohomology of endomorphism bundle of a holomorphic vector bundle over a complex manifold.

The rest of the paper is arranged as follows. We review the definition of  $A_\infty$  structure and the construction of Merkulov in §1. In §2, we review the abstract Hodge theory of a differential graded algebra and show how it leads to the data in Merkulov's construction. Applications to de Rham complex and deformation complex are presented in §3.

### 1. MERKULOV'S CONSTRUCTION OF CONSTRUCTION OF $A_\infty$ ALGEBRA

Let  $A = \oplus A^n$  be a  $\mathbb{Z}$ -graded vector space. An  $A_\infty$  (algebra) structure on a vector space  $A$  is a sequence of linear maps  $m_k : A^{\otimes k} \rightarrow A$ ,  $k \geq 1$ ,  $\deg m_k = 2 - k$  satisfying a sequence of conditions:

$$\begin{aligned}
 (A1) \quad & m_1^2 = 0, \\
 (A2) \quad & m_1(m_2(a_1 \otimes a_2)) = m_2(m_1(a_1) \otimes a_2) + (-1)^{|a_1|} m_2(a_1 \otimes m_1(a_2)), \\
 (A3) \quad & m_2(m_2(a_1 \otimes a_2) \otimes a_3) - m_2(a_1 \otimes m_2(a_2 \otimes a_3)) \\
 & = m_3(m_1(a_1) \otimes a_2 \otimes a_3) + (-1)^{|a_1|} m_3(a_1 \otimes m_1(a_2) \otimes a_3) \\
 & + (-1)^{|a_1|+|a_2|} m_3(a_1 \otimes a_2 \otimes m_1(a_3)),
 \end{aligned}$$

and so on. The formula (A1) says that  $m_1$  is a differential. If one writes  $d = m_1$ , then  $(A, d)$  is a cochain complex. Conversely, given a cochain complex, we can regard it as an  $A_\infty$  algebra with  $m_k = 0$  for  $k \geq 2$ . Similarly, if we regard  $m_2$  as a multiplication on  $A$  and write  $a_1 \cdot a_2 = m_2(a_1 \otimes a_2)$ , then (A2) says that  $d$  is a derivation with respect to this multiplication. Furthermore, if  $m_3 = 0$ , then (A3) implies that the multiplication  $\cdot$  is associative, and hence  $(A, \cdot, d)$  is a *differential graded algebra*. Conversely, any differential graded algebra can be regarded as an  $A_\infty$  algebra with  $m_k = 0$ ,  $k \geq 3$ . In this paper, we will be interested in  $A_\infty$  algebras with  $m_1 = 0$ , which by (A3), are graded associative algebras with higher multiplications.

We now review Merkulov's construction. Let  $(V, d)$  be a differential graded algebra,  $W$  be a sub complex of  $(V, d)$ , i.e. vector subspace  $W \subset V$  invariant under  $d$ .  $W$  is *not* assumed to be a subalgebra of  $V$ . Instead assume that there exists an odd operator  $Q : V \rightarrow V$  such that for any  $v \in V$  the element  $(1 - [d, Q])v$  lies in the subspace  $W$ , where  $[\cdot, \cdot]$  is the supercommutator. Define a series of linear maps

$$\lambda_n : \otimes^n V \rightarrow V, \quad n \geq 2,$$

starting with

$$\lambda_2(v_1, v_2) := v_1 \cdot v_2$$

and then recursively, for  $n \geq 3$ ,

$$\begin{aligned}
 & \lambda_n(v_1, \dots, v_n) \\
 = & (-1)^{n-1} [Q\lambda_{n-1}(v_1, \dots, v_{n-1})] \cdot v_n - (-1)^{n|v_1|} v_1 \cdot [Q\lambda_{n-1}(v_2, \dots, v_n)] \\
 & - \sum_{\substack{k+l=n+1 \\ k, l \geq 2}} (-1)^{k+(l-1)(|v_1|+\dots+|v_k|)} [Q\lambda_k(v_1, \dots, v_k)] \cdot [Q\lambda_l(v_{k+1}, \dots, v_n)].
 \end{aligned}$$

**Theorem 1.1.** (Merkulov [6]) *Let  $(V, d)$  be a differential graded algebra and  $(W, d) \subset (V, d)$  be a subcomplex as above. Then the linear maps*

$$m_k : \otimes^k W \rightarrow W, \quad k \geq 1,$$

*defined by*

$$(1) \quad \begin{aligned} m_1 &:= d, \\ m_k &:= (1 - [d, Q])\lambda_k, \quad \text{for } k \geq 2, \end{aligned}$$

*with  $\lambda_k$  being given above, satisfy the higher order associativity identities. I.e.,  $m_1, m_2, \dots$  define an  $A_\infty$  structure on  $W$ .*

## 2. ABSTRACT HODGE THEORY

Suppose that a differential graded algebra  $(A, \wedge, \mathfrak{d})$  is given a (Euclidean or Hermitian) metric  $\langle \cdot, \cdot \rangle$ , such that  $\mathfrak{d}$  has a formal adjoint  $\mathfrak{d}^*$ , i.e.,

$$\langle \mathfrak{d}a, b \rangle = \langle a, \mathfrak{d}^*b \rangle.$$

Since  $\mathfrak{d}^2 = 0$ , it follows that  $(\mathfrak{d}^*)^2 = 0$ . Set  $\square_{\mathfrak{d}} = \mathfrak{d}\mathfrak{d}^* + \mathfrak{d}^*\mathfrak{d}$ , and  $\mathcal{H} = \text{Ker } \square_{\mathfrak{d}} = \{a \in A : \square_{\mathfrak{d}}a = 0\}$ . Then  $\mathcal{H} = \{a : \mathfrak{d}a = 0, \mathfrak{d}^*a = 0\} = \text{Ker } \mathfrak{d} \cap \text{Ker } \mathfrak{d}^*$ . Assume that  $A$  admits a ‘‘Hodge decomposition’’:

$$A = \mathcal{H} \oplus \text{Im } \mathfrak{d} \oplus \text{Im } \mathfrak{d}^*.$$

It is standard to see that  $\text{Ker } \mathfrak{d} = \mathcal{H} \oplus \text{Im } \mathfrak{d}$ , and hence  $H(A, \mathfrak{d}) \cong \mathcal{H}$ . Notice that  $\square_{\mathfrak{d}}|_{\text{Im } \mathfrak{d} \oplus \text{Im } \mathfrak{d}^*}$  is invertible. Denote its inverse by  $\square^{-1}$ . Consider the Green’s operator  $G : A \rightarrow A$ , which is defined as the composition

$$A \rightarrow \text{Im } \mathfrak{d} \oplus \text{Im } \mathfrak{d}^* \xrightarrow{\square^{-1}} \text{Im } \mathfrak{d} \oplus \text{Im } \mathfrak{d}^* \rightarrow A,$$

where the first arrow is the inclusion, and the last arrow is the projection. Since  $\square_{\mathfrak{d}}$  commutes with  $\mathfrak{d}$  and  $\mathfrak{d}^*$ , so does  $G$ . For any  $\alpha \in A$ , denote  $f(\alpha)$  by  $\alpha^H$ . Then we have

$$(2) \quad \alpha - \alpha^H = \square_{\mathfrak{d}} G \alpha = (\mathfrak{d}\mathfrak{d}^* + \mathfrak{d}^*\mathfrak{d})G \alpha = \mathfrak{d}\mathfrak{d}^* G \alpha + \mathfrak{d}^* G \mathfrak{d} \alpha.$$

This gives the explicit Hodge decomposition. Note that there is an induced wedge product on  $H(A, \mathfrak{d})$ .

Hence  $\mathcal{H}$  has an induced product by virtue of being isomorphic to  $H(A, \mathfrak{d})$  as vector spaces. However, in general the product of two elements in  $\mathcal{H}$  may not lie in  $\mathcal{H}$ . For  $\alpha, \beta \in \mathcal{H}$ , define

$$\alpha \circ \beta = (\alpha \wedge \beta)^H.$$

Merkulov [6] made the remark that this multiplication is in general not associative. The following lemma shows that his remark is incorrect.

**Lemma 2.1.** *For  $\alpha, \beta, \gamma \in \mathcal{H}$ , we have*

$$((\alpha \wedge \beta)^H \wedge \gamma)^H = (\alpha \wedge \beta \wedge \gamma)^H.$$

*Hence  $(\mathcal{H}, \circ)$  is associative.*

*Proof.* By (2), we have

$$\begin{aligned} (\alpha \wedge \beta)^H &= \alpha \wedge \beta - \mathfrak{d}\mathfrak{d}^* G(\alpha \wedge \beta) + \mathfrak{d}^* G \mathfrak{d}(\alpha \wedge \beta) \\ &= \alpha \wedge \beta - \mathfrak{d}\mathfrak{d}^* G(\alpha \wedge \beta). \end{aligned}$$

Therefore,

$$\begin{aligned} (\alpha \wedge \beta)^H \wedge \gamma &= \alpha \wedge \beta \wedge \gamma - \mathfrak{d}\mathfrak{d}^*G_{\mathfrak{d}}(\alpha \wedge \beta) \wedge \gamma \\ &= \alpha \wedge \beta \wedge \gamma - \mathfrak{d}(\mathfrak{d}^*G_{\mathfrak{d}}(\alpha \wedge \beta) \wedge \gamma). \end{aligned}$$

Hence

$$((\alpha \wedge \beta)^H \wedge \gamma)^H = (\alpha \wedge \beta \wedge \gamma)^H - (\mathfrak{d}(\mathfrak{d}^*G_{\mathfrak{d}}(\alpha \wedge \beta) \wedge \gamma))^H = (\alpha \wedge \beta \wedge \gamma)^H.$$

As a consequence,

$$\begin{aligned} (\alpha \circ \beta) \circ \gamma &= ((\alpha \wedge \beta)^H \wedge \gamma)^H = (\alpha \wedge \beta \wedge \gamma)^H \\ &= (\alpha \wedge (\beta \wedge \gamma))^H = \alpha \circ (\beta \circ \gamma). \end{aligned}$$

□

In fact, another to check the associativity is the following

**Lemma 2.2.** *The isomorphism  $\phi : \mathcal{H} \rightarrow H(A, \mathfrak{d})$  given by  $\alpha \mapsto [\alpha]$  (where  $[\alpha]$  denote the cohomology class of  $\alpha$ ) is a ring isomorphism  $(\mathcal{H}, \circ) \rightarrow (H(A, \mathfrak{d}), \wedge)$ .*

*Proof.* For  $\alpha, \beta \in \mathcal{H}$ , we have

$$\phi(\alpha \circ \beta) = [(\alpha \wedge \beta)^H] = [\alpha \wedge \beta] = [\alpha] \wedge [\beta] = \phi(\alpha) \wedge \phi(\beta).$$

□

Our main observation is that it is possible construct an  $A_{\infty}$  structure on  $\mathcal{H}$  by the method of Merkulov [6] reviewed in last section. Indeed, let  $V = A$ ,  $W = \mathcal{H}$ , and  $Q = G\mathfrak{d}^*$ . Then (2) implies that  $(W \subset V, \mathfrak{d}, Q)$  satisfies the conditions in last section. By Theorem 1.1, we get

**Theorem 2.1.** *For a differential graded algebra  $(A, \wedge, \mathfrak{d})$  with a Euclidean or Hermitian metric, such that  $\mathfrak{d}$  has a formal adjoint  $\mathfrak{d}^*$  and  $A$  has a Hodge decomposition*

$$A = \mathcal{H} \oplus \text{Im } \mathfrak{d} \oplus \text{Im } \mathfrak{d}^*,$$

*where  $\mathcal{H} = \text{Ker } \square_{\mathfrak{d}}$ , there is a canonical  $A_{\infty}$  structure on  $\mathcal{H}$  with  $m_2 = \circ$ .*

By Lemma 2.2, we get

**Theorem 2.2.** *For a differential graded algebra  $(A, \wedge, \mathfrak{d})$  with a Euclidean or Hermitian metric, such that  $\mathfrak{d}$  has a formal adjoint  $\mathfrak{d}^*$  and  $A$  has a Hodge decomposition*

$$A = \mathcal{H} \oplus \text{Im } \mathfrak{d} \oplus \text{Im } \mathfrak{d}^*,$$

*there is a canonical  $A_{\infty}$  structure on  $H(A, \mathfrak{d})$  with  $m_2$  the induced wedge product.*

### 3. APPLICATIONS

**3.1. The Riemannian case.** Let  $(M, g)$  be an oriented closed Riemannian manifold. Then there is an induced Euclidean metric on  $\Omega^*(M)$ . Let  $\Omega^*(M)$  be the space of differential forms on  $M$ , it has a wedge product  $\wedge$  and the exterior differential  $d$ , such that  $(\Omega^*(M), \wedge, d)$  is a differential graded algebra. We call it the *de Rham algebra*. Using the orientation, one can define the Hodge star operator  $*$ . The formal adjoint of  $d$  is given by

$$d^* = - * d *.$$

Let  $\square = dd^* + d^*d$ . Then  $\square$  is a second order elliptic operator, and standard elliptic operator theory gives the Hodge decomposition

$$\Omega^*(X) = \mathcal{H} \oplus \text{Im } d \oplus \text{Im } d^*,$$

where  $\mathcal{H}$  is the space of harmonic forms. Theorem 2.2 then yields a canonical  $A_\infty$  structure on the de Rham cohomology  $H^*(M) = H^*(\Omega^*(M), d)$ .

**3.2. The oomplex case.** Let  $W$  be a closed complex manifold,

$$\begin{aligned}\Omega^{*,*}(W) &= \oplus_{p,q \geq 0} \Gamma(W, \Lambda^p T^* W \otimes \Lambda^q \overline{T}^* W), \\ \Omega^{-*,*}(W) &= \oplus_{p,q \geq 0} \Gamma(W, \Lambda^p T W \otimes \Lambda^q \overline{T}^* W).\end{aligned}$$

On both of these spaces, there is a wedge product  $\wedge$  and a  $\bar{\partial}$  operator, which form differential graded algebras. We call them *Dolbeault algebra* and *deformation algebra* respectively. Their cohomologies are denoted by  $H^{*,*}(W)$  and  $H^{-*,*}(W)$  respectively. Given a Hermitian metric on the holomorphic tangent bundle  $TW$ , there are induced Hermitian metrics on both  $\Omega^{*,*}(W)$  and  $\Omega^{-*,*}(W)$ . It is easy to find the formal adjoints of the operators  $\bar{\partial}$  on these spaces and define the corresponding Laplacian operators. Again we get elliptic operators and by standard theory the Hodge decompositions. Therefore, we have canonical  $A_\infty$  structures on both  $H^{*,*}(W)$  and  $H^{-*,*}(W)$ .

**3.3. The bundle case.** Let  $X$  be a closed complex manifold, and  $\pi : E \rightarrow X$  a holomorphic vector bundle. Consider the holomorphic vector bundle  $\text{End}(E) \rightarrow X$  and the space  $\mathcal{A} = \Omega^{0,*}(\text{End}(E)) = \Gamma(X, \Lambda^* \overline{T}^* X \otimes \text{End}(E))$ . There is an induced wedge product on  $\mathcal{A}$ , which is in general not graded commutative. The  $\bar{\partial}_{\text{End}(E)}$  operator for  $\text{End}(E)$  gives a differential for  $(\mathcal{A}, \wedge)$ . Given a Hermitian metric on  $X$  and a Hermitian metric on  $E$ , one can consider the formal adjoint operator  $\bar{\partial}_{\text{End}(E)}^*$ . The corresponding Laplacian operator is elliptic, hence we have Hodge theory. Therefore, we have an induced  $A_\infty$  structure on the Dolbeault cohomology  $H^*(X, \text{End}(E))$ .

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